SYMMETRY, BIFURCATIONS AND PATTERN FORMATION (d' apres Sattinger, Michel, Thom and many others)

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1. INTRODUCTION

Nature often seems to like (approximately) symmetric solutions to problems. Mathematically, or more generally, scientifically, it thus becomes our task to understand why, e.g. by showing that more or less regular patterns are usually the most stable, or economical, or optimising with respect to a suitable criterion.

Particular aspects of this theme concern the question of whether symmetric problems necessarily have symmetric solutions and how the symmetry of a problem/solution can change as a parameter varies. This last problem in turn is part of bifurcation theory which examines the question of how the set of solutions of a problem can change in nature as certain parameters vary. Here it turns out to be remarkably fruitful [9, 29, 33] to take the symmetry of a situation into account. Also the presence of a symmetry group has remarkably strong consequences for extrema of functions as we shall see.

There are two aspects of symmetry which I consider very interesting and which perhaps so far have not had all the attention they deserve. One concerns approximate symmetry. Here I have in mind for instance a symmetric problem which would have a symmetric solution if the boundary conditions were equally symmetric. Now suppose the boundary conditions are disturbed, when will there be an approximatedly symmetric solution, e.g. when will there be a boundary layer in which the symmetry will be restored and a fully symmetric solution in the middle or when will the solution of the new problem be like a crystal with defects. There seems to be no general theorem concerning such matters. Other approximate symmetry problems (e.g. how to

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recognise them) arise when dealing with a basically symmetric pattern with a (small) random [3] or systematic disturbance superimposed. Some preliminary foundational remarks on approximate symmetry are contained in [17, 28]. Much more remains to be done and different kinds of approximate symmetry certainly exist. Still harder to understand approximatedly symmetric entities are the Penrose universes described in section 2 below. The second aspect concerns the matter that as certain parameters change an object may both lose and gain symmetry, sometimes simultaneously. The systematics of symmetry loss, that is spontaneous breaking of symmetry, have had considerable attention over the years, see e.g. [17, 20, 21, 22, 24]. The systematics of gaining symmetry far less. The matter is discussed below in section 3 in terms of the automorphisms of three dimensional algebras.

Apart from this example there is little new in this paper and it should basically be seen as a low key introduction to important work of others, notably the persons mentioned in the title. I have added a few more references than are strictly necessary for the purposes of this paper itself, e.g. a few references to books which treat of bifurcation theory and give applications [32, 25, 18, 26, 11, 1] and a few (in my view) closely related matters [2, 10, 41, 23].

2. PATTERN FORMATION AND SYMMETRY AND APPROXIMATE PATTERNS AND SYMMETRIES

One of the best known examples of pattern formation and one of the most studied (the two terms are not synonymous) is the Benard convection. Consider a fluid layer heated from below as in Figure 1. At a small temperature difference (small temperature gradient) heat is transported by conduction and the solution

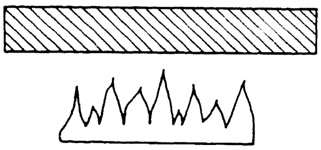


Figure 1. Fluid layer heated from below. After [12]

is completely symmetric, that is if we consider only the possible symmetrics viewable from the top we have E_2 symmetry where E_2 is the group of rigid motions of the plane (the fluid layer is assumed to be infinitely extended). At higher temperature gradients the lighter fluid at the bottom will tend to rise, cool at the top and return to the bottom. A microscopic pattern arises which can take the form of rolls, c.f. Figure 2 or the famous Bénard cells (hexagons), c.f. Figure 3.

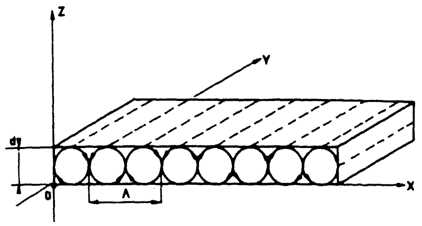


Figure 2. Rolls pattern. After [12]

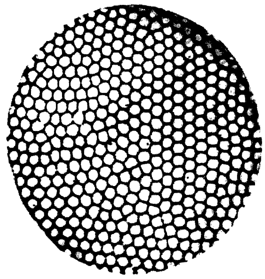


Figure 3. Hexagonal pattern. After [5]

Other less regular patterns develop at still higher temperature gradients. We will say a little more about Sattinger's bifurcation-in-the-presence-of-symmetry analysis of the Bénard problem in section 5 below. A highly recommended up to date discussion of the topic is [9], which includes also a discussion of the spherical case, a model for convection in the molten layer of the earth between core and mantle.

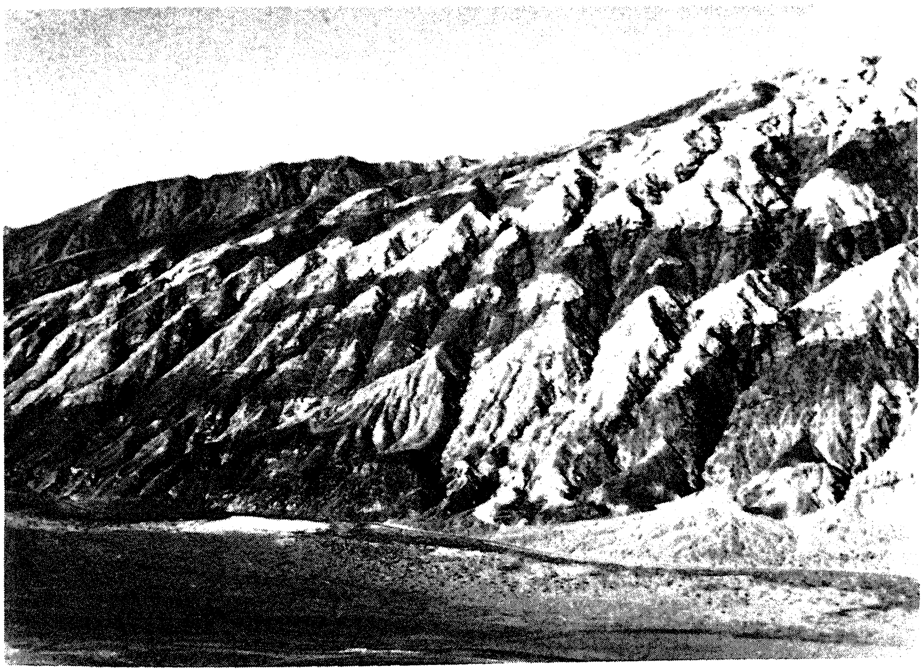


Figure 4. East wall of Death Valley, California. After [35].

Another example of a strikingly regular pattern, caused by erosion in this case, is depicted in Figure 4 and still more examples are the drainage basin patterns of the Figures 5, 6, and 7 below. These drainage patterns seem perhaps less regular

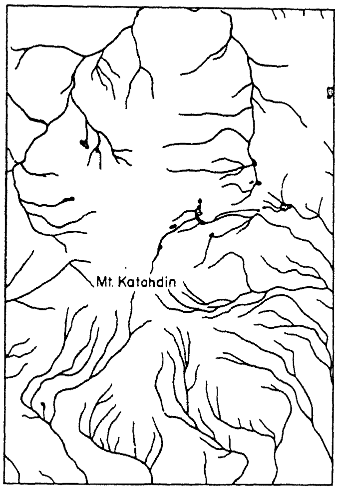


Figure 5. Radial drainage pattern. After [39]

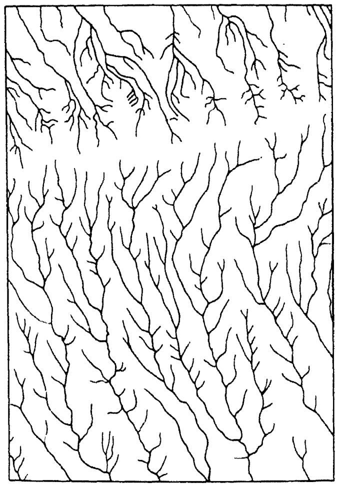


Figure 6. Parallel drainage pattern. After [39]

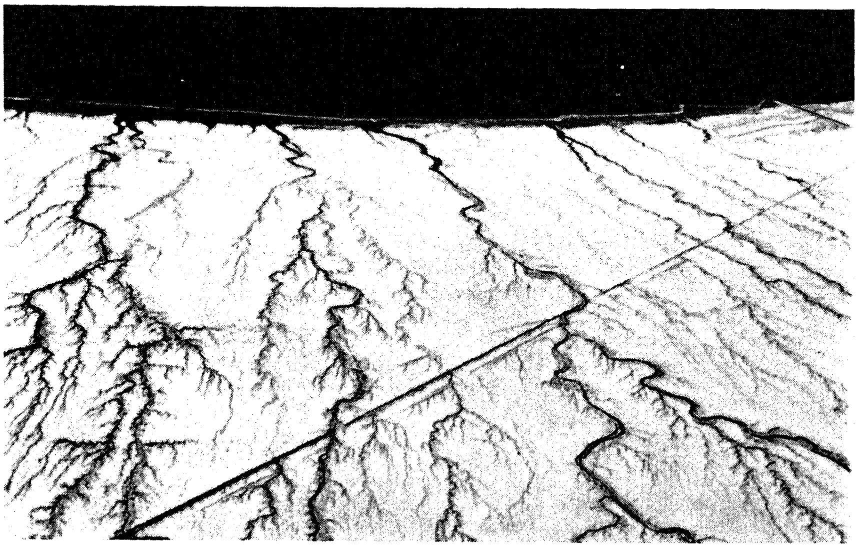


Figure 7. Dendritic channels. After [35].

than the erosion pattern of Figure 4 and the convection patterns of the Figures 2 and 3. Still they are very far from random and as such demand an explanation of their (amount of) regularity. In all these examples the problem is not so much to understand that something happens, i.e. that some pattern develops, but to explain the striking regularity of the patterns and in case there are several patterns to understand the selection mechanisms and the relative stability of these patterns with respect to each other.

Still other examples of spontaneously arising regular patterns are the socalled cloud streets, c.f. Figure 8 below, and the socalled Liesegang rings, which form e.g. when a drop of silver nitrate is placed on a film of gelatine saturated with potassum dichromate as in Figure 9.

However, both nature and man seem to like approximately symmetric solutions even better. Or solutions whose obvious regularity is much harder to describe in mathematical terms then e.g. hexagonal or street patterns. Spiral patterns for instance, occur very frequently, c.f. Figures 10 and 11 below

Figure 8. Stratocumulus cloud streets. After [37]

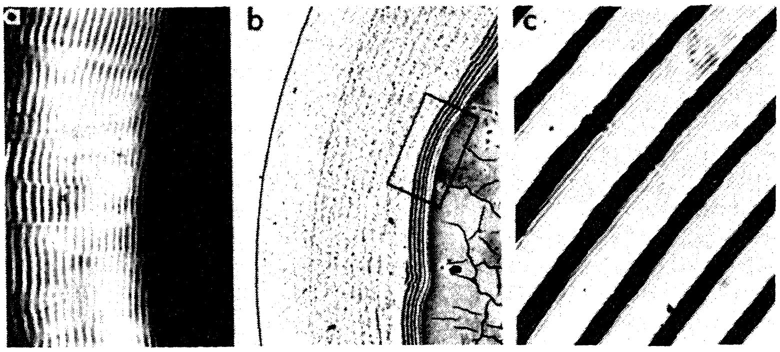


Figure 9. Liesegang rings. After [4]

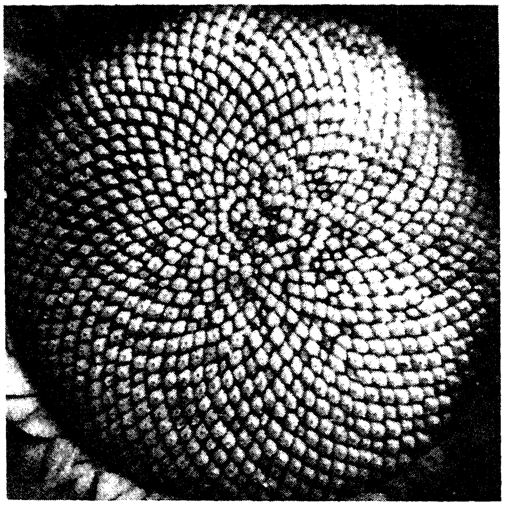


Figure 10. Sunflower head. Courtesy of Empire magazine

Interesting remarks on the mathematics of spirals can be found in for instance [6]. Still harder to describe kinds of symmetry are those exhibited by various fractal patterns such as the twin dragon pattern of Figures 12 and 13 below which not only have certain more or less obvious symmetries (c.f. Figure 13) and the less obvious rotational symmetry of Figure 12, also has the property that it can be covered with reduced size replicas of itself ad infinitum.

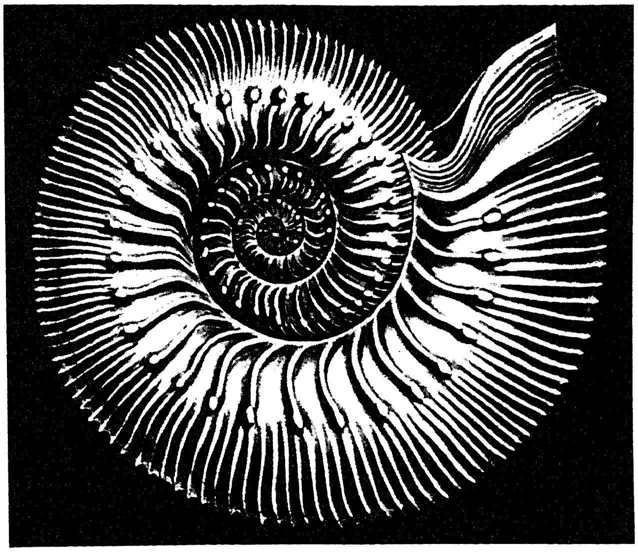


Figure 11. After [36]

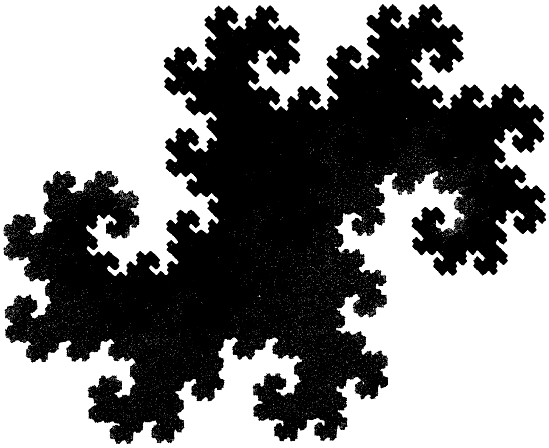


Figure 12. Twin dragon fractal, After B.B. Mandelbrot, The fractal geometry of nature, Freeman, 1982

Consider also the spiral tiling of H. Voderberg depicted in Figure 14. Though it obviously has many regularity properties and is intuitively very symmetric it is quite difficult to find symmetries (apart from a 180 degree rotation which also interchanges colours).

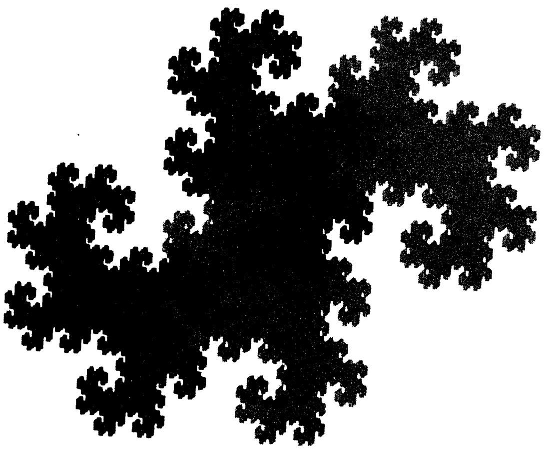


Figure 13. Tiling of twin dragon fractal, After B.B. Mandelbrot, The fractal geometry of nature, Freeman, 1982

Figure 14. A non-periodic tiling by H. Voderberg. From [8]

There are patterns whose "regularity" is even more disturbing and very hard, possibly impossible to describe in the mathematical framework we usually use in this connection. These are the Penrose non-periodic tilings, also called the Penrose universes as described in [8] from which the following is taken.

The basic tiles are obtained from a diamond with angles of 72° and 108° as drawn in Figure 15. The number t is the golden ratio $\frac{1}{2}(1 + \sqrt{5})$. The two tiles, called dart and kite, so obtained are marked with a drawn and dashed circle as indicated and there is an additional tiling rule in that abutting edges must join circle segments of the same kind so that fitting a dart and a kite to form a diamond is forbidden. Using these two kinds

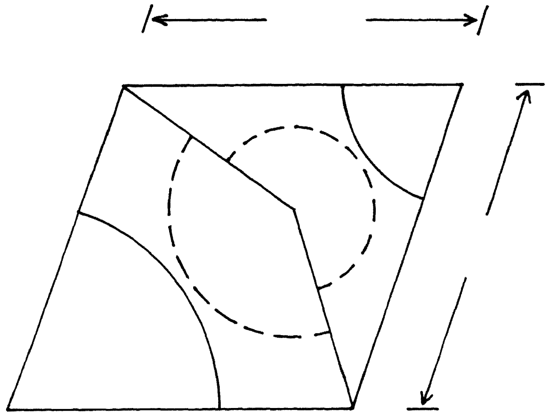


Figure 15. The Penrose dart and kite.

of tiles and respecting the additional tiling rules it is possible to tile the entire plane. Indeed there are innumerable different ways of doing that. Some of the more striking patterns are depicted below in the Figures 16, 17 and 18. The central 10-sided regular polygon consisting of 15 darts and 25 kites in Figure 18 is called a cartwheel. Note also that the cartwheel pattern of Figure 18 has little symmetry in the obvious sense (only a reflection through a central vertical line).

Here are some properties of the Penrose tilings (or universes):

a) all tilings are non-periodic;
b) every point in every tiling is inside a cartwheel;
c) every finite region of diameter $\leq$ d of any tiling occurs within distance 2d of any point in any other tiling.

In addition there are "fractal properties" in that from a given tiling another one with larger kites and darts can be constructed in a simple systematic way.

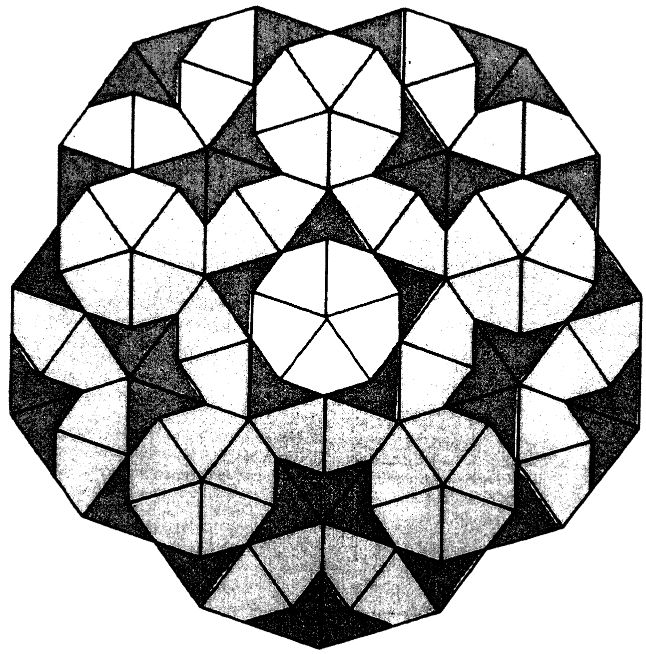


Figure 16

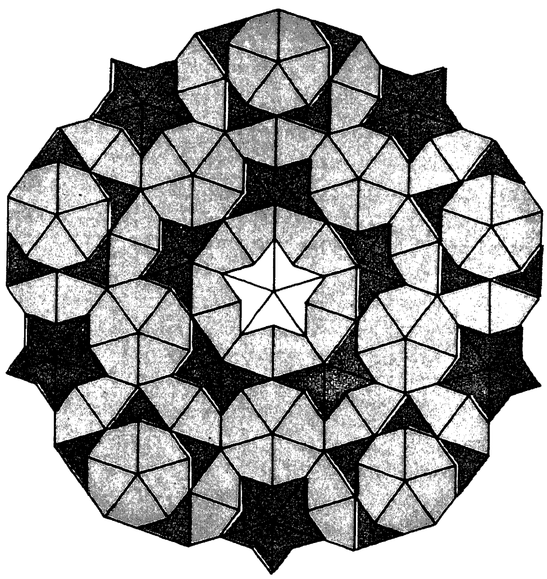


Figure 17

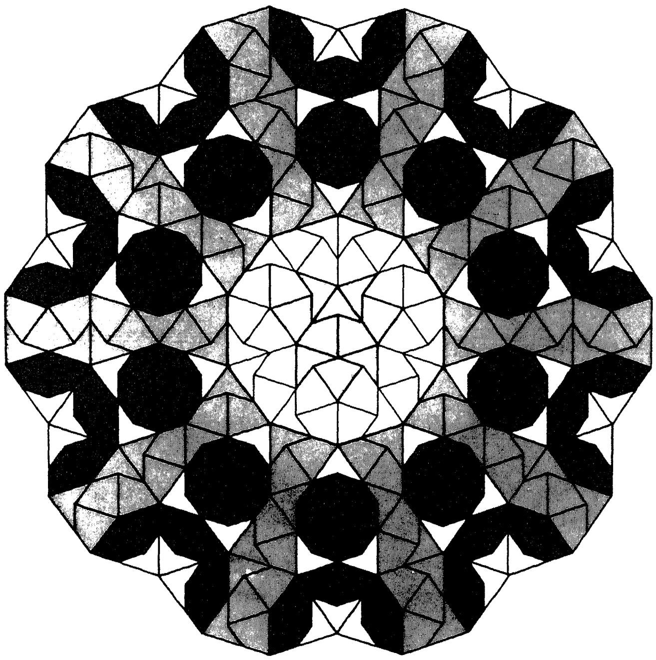


Figure 18

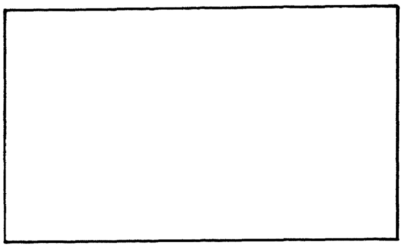
Properties b) and c) above are regularity, indeed symmetry, properties especially property c) when applied to the same tiling. The "fractal property" is also a symmetry property of course. Yet these symmetries are of a different kind then what we usually understand by the word.

2. CHANGES IN SYMMETRY AS A PARAMETER VARIES

This section contains a number of remarks pertaining to what can happen to the symmetry group of an object as a parameter varies, i.e. during a deformation.

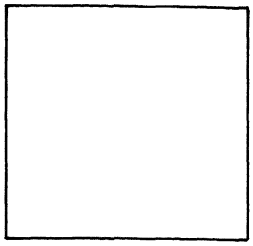
2.1. Example

Consider a rectangle with sides 1 and λ For $\lambda \neq 1$, the symmetry group is generated by the two reflections across the central horizontal and vertical axis, so that the symmetry group is the Klein four group $V_4 = Z/(2) \times Z/(2)$. For $\lambda = 1$ however, there suddenly appears an additional bit of symmetry, viz a rotation through 90°. For this value of λ the symmetry group is suddenly larger.



Symmetry group Z/(2) x Z/(2)

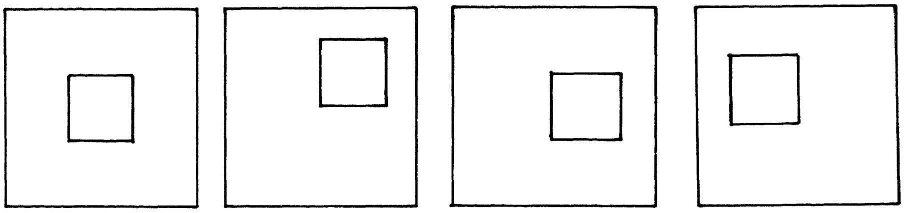
Figure 19



Symmetry group D_4

2.2. Example

Consider a small square inside a larger one with the sides of the small square parallel to those of the big one as in Figure 20. The parameter which varies is the position of the smaller square



(a) (b) (c) (d)

Figure 20

If the centre of the small square is precisely in the middle of the big one the symmetry group of the whole figure is D_4 (Figure 20 (a)), if the centre is on a diagonal but not in the centre the only non-trivial symmetry is a reflection across that diagonal (Figure 20 (b)). The situation is analogus for the centre on the horizontal or vertical symmetry axis of the big square (Figure 20 (c)) and finally if the centre is on none of these four lines there is no non-trivial symmetry. Thus, in the parameter square of this example we can describe the symmetry of the various figures as in Figure 21 below. A similar picture holds for example 2.1. And indeed in certain reasonably general situations one can show that this is the general pattern as we shall see below in section 2.3. Though of course it may happen that at a certain critical value the symmetry group increases by an infinite amount as when one considers the rigid

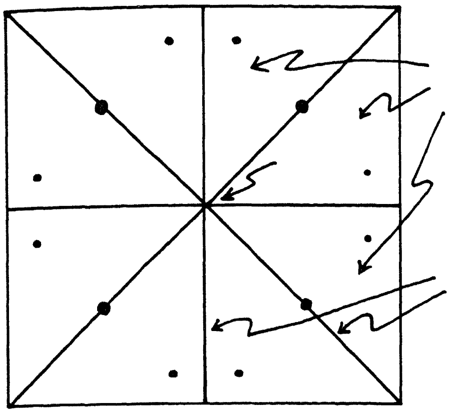


Figure 21

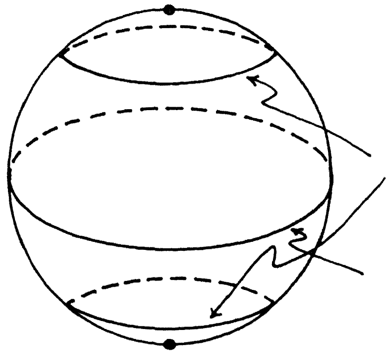


Figure 22

motions of the plane which take a given ellipse with major axis λ and minor axis 1 into itself. When λ becomes one (circle) the symmetry group suddenly increases from $Z/(2) \times Z/(2)$ to the circle group S^1 of all rotations about the centre of the ellipse.

2.3. A general mathematical framework for symmetry breaking [20, 21]

We can view the examples above as follows. There is a group of potential symmetries, in this case the rigid motions of the plane. For certain isolated parameter values a "large" subgroup of this group defines actual symmetries and as the parameter moves away from this value the symmetry is broken to a smaller subgroup.

A more precise and mathematical setting for this picture of symmetry breaking is as follows.

Let G be a group acting on a set M. (I.e. there is given a map $G \times M \to M$, written $(g,m) = gm$ such that $g_1(g_2m) = (g_1g_2)m$ $1m = m$; think of gm as the result of applying the transformation g to m). For instance in the case of example 2.2. we would have $G = E_2$, the group of rigid motions of the plane, and the set M is the set of all squares of sizes 6 and 2, the latter inside the other with sides parallel to the co-ordinate axes of the plane. M can be conveniently described as M $= \{(P_1,P_2)\ \ R^2 \times R^2 : d(P_1,P_2) \leqslant 2\}$ (P_1 is the center of the first square, P_2 that of the second).

For each $m \in M$, the isotropy subgroup at m is

$$G_m = \{g \in G : gm = m\} \qquad (2.1)$$

This is the group of symmetries of the structure represented by m M. The orbit of $m \in M$ is

$$G_m = \{gm : g \in G\} \tag{2.2}$$

G acts transitively on Gm (i.e. for every $x, y \in Gm$ there is a $g \in G$ such that $gx = y$). A set with an action of G on it is called a G-set. Two G-sets are isomorphic if there is a bijection $\phi: M_1 \to M_2$ such that $\phi(gm_1) = g\phi(m_1)$ for all $m_1 \in M_1$. If M is a transitive G-set the isotropy subgroups of the points of M are all conjugate and this sets up a bijection between isomorphism classes of transitive G-sets and conjugacy classes of subgroups of G.

A stratum of G is the union of all orbits belonging to one isoclass of G-sets.

Consider for example $M = S^2$, the sphere, and $G = O_2(R)$ the group of all rotations around the N-S axis and inversion through the origin. An orbit is then the union of two parallel circles at equal northern and southern latitude. There are three strata viz $\{N\} \cup \{S\}$ and the equator, both are strata consisting of a single orbit, and the third stratum is the union of all other orbits. A precisely similar picture is offered by the description of example 2.2. by means of Figure 21. By restricting immediately to the subgroup D_4 of E_2 which leaves the larger square invariant we have $G = D_4$ and M is a square of side 4.

Orbits are e.g. sets of 8 points as indicated by dots in Figure 21, or sets of four points as indicated by crosses, or sets of four points as indicated by small circles and finally the centre is an orbit. There are four strata given precisely by these four types of orbits.

Returning to the general situation. If G is a compact Lie group acting smoothly on a smooth manifold M everything is beautiful: the isotropy subgroups are closed Lie subgroups, orbits and strata are submanifolds and there exists a G-invariant Riemannian metric on M. A consequence of all this is:

2.4. Theorem

Let G be a compact Lie group acting smoothly on a smooth manifold M. Then for every $m \in M$ there is a neighbourhood U of m such that G_m is larger than $G_{m'}$ (up to conjugacy) for all $m' \in M$.

Thus symmetry can suddenly decrease but not suddenly increase which is precisely as in examples 2.1. and 2.2. and also as in the example of Figure 22.

Note that in this setting hidden symmetry may occur. To see this consider the following modification of the example of Figure 22. Consider again S^2 and let the group G now consist of rotations around the N-S axis and reflexion through the equator plane. For M take the space of all unordered pairs of parallels. There is an obvious induced action of G on M by viewing G as a group of transformations on S^2 so that each element of G takes an element of M to a possibly different element of M. Let P be the element of M consisting of twice the equator. The isotropy of P is all of G. The rotations are visible. However, the reflection (when restricted to the submanifold of S^2 represented by P) acts just like the identity. That is as a symmetry of the figure P in S^2 it is a hidden symmetry, which appears as soon as P is deformed into a pair of close together equal latitude north and south parallels.

2.5. Example

Consider the group of all transformations $R^2 \to R^2$ of the plane into itself of the form $(f,g) : (x,y) \to (f(x),\ y + g(x))$ where $f: R \to R$ is a diffeomorphism and $g: R \to R$ is any differentiable map. The inverse of (f,g) is the element $(f^{-1}, -gf^{-1})$ and the composition goes as follows: $(f_1,g_1) \circ (f_1,g_1) = (f_2f_1,\ g_1 + g_2f_1)$. The identity element is (id, o). Thus G is a subgroup of the group $Diff(R^1)$ of all diffeomorphisms of R^2 into itself. Now let M be the space of all unordered pairs of elements of R^2. Consider an element $P = \{x_1,y_1),\ (x_2,y_2)\} \in M$ and let us calculate the isotrophy subgroup of P. Suppose $\phi \in G$ is in G_p. Then we must have

$$\phi(x_1,y_1) = (x_1,y_1) \text{ and } \phi(x_2,y_2) = (x_2,y_2) \qquad \text{(i)}$$

or

$$\phi(x_1,y_1) = (x_2,y_2) \text{ and } \phi(x_2,y_2) = (x_1,y_1) \qquad \text{(ii)}$$

or both which can only happen if $(x_1,y_1) = (x_2,y_2)$ It is quite easy to describe the isotropy subgroup for all P. For the purposes of this example, however, we need only two cases.

$x_1 = x_2$ and $y_1 \neq y_2$. (a)

Then $G_p = \{(f,g) : f(x_1) = x_1, g(x_1) = 0\}$

$x_1 \neq x_2$ and $y_1 \neq y_2$

Then $G_p = \{(f,g) : f(x_1) = x_1,\ f(x_2) = x_2,\ g(x_1) =$ (b)

$= g(x_2) = 0\} \cup$

$\{(f,g) : f(x_1) = x_2,\ f(x_2) = x_1,\ g(x_1) = y_2 - y_1,$

$g(x_2) = y_1 - y_2\}$

In case (b) the set

$N_p = \{(f,g) : f(x_1) = x_1,\ f(x_2) = x_2,\ g(x_1) = g(x_2) = 0\}$ is a normal subgroup of G_p and G_p is in fact the disjoint union N_p αN_p where $\alpha = (f,g)$ is any element of G such that

$$f(x_1) = x_2,\ f(x_2) = x_1\ ,\ g(x_1) = y_2 - y_1\ ,\ g(x_2) = y_1 - y_2.$$

Now let $P = ((x_1,y_1), (x_2,y_2))$ approach a point Q $=((x,y_1), (x,y_2))$ with $y_1 \neq y_2$. Then we see that as the limit point is reached there is both symmetry gain in that the N_p part of G_p becomes bigger and sudden symmetry loss in that the αN_p part of G_p disappears. It is in fact easy to show that there are no inclusion up to conjugacy relations between G_p and G_Q.

Thus theorem 2.6. does not hold in more general situations. It seems that this kind of phenomenon cannot happen when considering the symmetry group (of motions) of figures or patterns in Eudidean space as these figures change. But I do not know of a general theorem to this effect.

3. DESIGN SYMMETRY VERSUS GENERIC SYMMETRY [14]

In example 2.7. we saw that in more general cases a statement like that of theorem 2.6 does not hold. A more complicated but also more suggestive example of the same phenomenon of simultaneous symmetry loss and symmetry gain during a deformation is obtained by considering the automorphisms of three (or higher) dimensional algebras over R. That is the subject of this section.

3.1. Algebra structures

Let $V = R^3$ be the vectorspace of all 3-tuples of real numbers. An associative algebra structure with unit on V is given by a bilinear map (the multiplication).

$$m: V \times V \to V,\ (x,y) \to xy$$

such that $(xy)z = x(yz)$ and such that there exists a $1 \in V$ with $1x = x1 = x$ for all $x \in V$. By choosing a basis in V suitably we can assume that $1 = e_1 = (1,0,0)$ and we shall do so. Let e_2, e_3 be the other basis elements. Then because of the bilinearity the multiplication is specified by 12 constants (the so-called structure constants)

$$e_2e_2 = \gamma_{22}^1 e_1 + \gamma_{22}^2 e_2 + \gamma_{22}^3 e_3 \qquad e_2e_3 = \gamma_{23}^1 e_1 + \gamma_{23}^2 e_2 + \gamma_{23}^3 e_3$$

$$e_3e_2 = \gamma_{32}^1 e_1 + \gamma_{32}^2 e_2 + \gamma_{32}^3 e_3 \qquad e_3e_3 = \gamma_{33}^1 e_1 + \gamma_{33}^2 e_2 + \gamma_{33}^3 e_3$$

In order that the algebra be associative these γ_{ij}^k have to satisfy certain relations. E.g. from $e_2(e_2e_3) = (e_2e_2)e_3$ one obtains

$$\begin{aligned} \gamma_{23}^2\gamma_{22}^1 + \gamma_{23}^3\gamma_{23}^1 &= \gamma_{22}^2\gamma_{23}^1 + \gamma_{22}^3\gamma_{33}^1 \\ \gamma_{23}^1 + \gamma_{23}^3\gamma_{23}^2 &= \gamma_{22}^3\gamma_{33}^2 \\ \gamma_{23}^2\gamma_{22}^3 + \gamma_{23}^3\gamma_{23}^3 &= \gamma_{22}^1 + \gamma_{22}^1\gamma_{23}^3 + \gamma_{22}^3\gamma_{33}^3 \end{aligned} \tag{3.1}$$

The precise form of these conditions will not be important for us.

3.2. Isomorphisms and automorphisms

A map ϕ: A → B from an algebra A to an algebra B is an isomorphism if it is an isomorphism of vectorspaces and if moreover $\phi(1_A) = 1_B$ and $\phi(xy) = \phi(x)\ \phi(y)$ for all $x,y \in A$. Here 1_A and 1_B denote the unit elements of A and B. An isomorphism ϕ: A → A is called an automorphism and Aut(A) denotes the group of automorphisms of A. This group should also be considered as the group of symmetries of the algebra A. Indeed the situation fits the general framework described above. Let G be the group of all vector space isomorphisms ϕ: V → V such that $\phi(e_1) = e_1$ (because we only consider algebra structures on V for which e_1 is the unit element). Let M be the space of all 12-tuples $(\gamma_{22}^1, \ldots, \gamma_{33}^3)$ such that all associativity relations like (3.1) hold. For ϕ: V → V, $\phi(e_1) = e_1$, let $\bar{e}_2 = \phi(e_2)$, $\bar{e}_3 = \phi(e_3)$ and let $\bar{\gamma}_{22}^1, \ldots, \bar{\gamma}_{33}^3$ be defined by

$$\bar{e}_i \cdot \bar{e}_j = \bar{\gamma}_{ij}^1 e_1 + \bar{\gamma}_{ij}^2 \bar{e}_2 + \bar{\gamma}_{ij}^3 \bar{e}_3 \quad , \ i,j = 2,3$$

The group element ϕ now acts on M by

$$(\gamma_{22}^1, \ldots, \gamma_{33}^3) \to (\bar{\gamma}_{22}^1, \ldots, \bar{\gamma}_{33}^3).$$

One sees immediately that if A is an associative algebra with structure constants $\gamma = (\gamma_{22}^1, \dots, \gamma_{33}^3)$ then $\mathrm{Aut}(A) = G_\phi$, the isotropy subgroup of γ M.

Two algebras A and B are isomorphic if and only if they (or more precisely the corresponding 12-tuples of structure constants) are in the same orbit.

3.3. The isomorphism classes of three dimensional algebras

It turns out that up to isomorphism there are six different algebra structures on $V = R^3$. They are

$A_1 \simeq R[X]/X(X-1)(X-2)$

$A_2 \simeq R[X]/X(X^2 + 1)$

$A_3 \simeq R[X]/X^3$

$A_4 \simeq R[X]/X^2(X-1)$

$A_5 \simeq R[X,Y]/(X^2, Y^2, XY)$

A_6 with basis $1, e_2, e_3$ and the multiplication rules
$e_2^2 = 1,\ e_3^2 = 0,\ e_2e_3 = e_3,\ e_3e_2 = -e_3$

Here if $f(X) = X^3 + a_2X^2 + a_1X + a_o$ is a polynomial $R[X]/f(X)$ denotes the 3 dimensional algebra with basis $1, X, X^2$ and multiplication rules $XX = X^2$, $XX^2 = X^2X = -a_2X^2 - a_1X - a_o$. The algebra A_5 is defined similarly.

3.4. The deformation/contraction relations between the six isomorphism classes

By the symbol $A \Rightarrow B$ we understand that there is a family of algebra structures $A(t) = (\gamma_{22}^1(t), \dots, \gamma_{33}^3(t))$ isomorphic for small $t \neq o$ to A and such that A(o) is isomorphic to B. In other words $A \Rightarrow B$ means that the orbit corresponding to the isomorphism class B is in the closure of the orbit corresponding to isomophism class A.

With this notation the pattern of contraction/deformation relations between $A_1, \dots, A_6$ is as follows

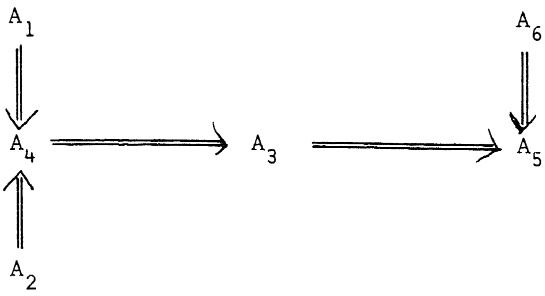


Figure 23

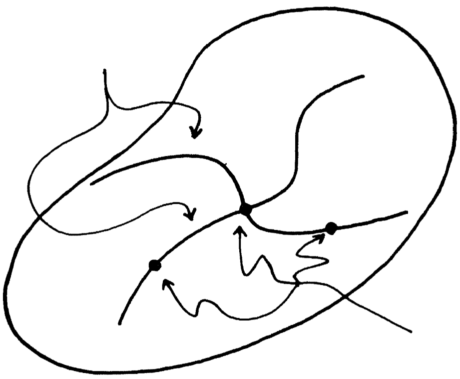


Figure 24

Concentrating on the A_1, A_3, A_4 part of Figure 23 the situation in γ-space is somewhat like depicted in Figure 24. Compare this e.g. with Figure 22 and Figure 21.

The corresponding pattern of automorphism groups is as follows

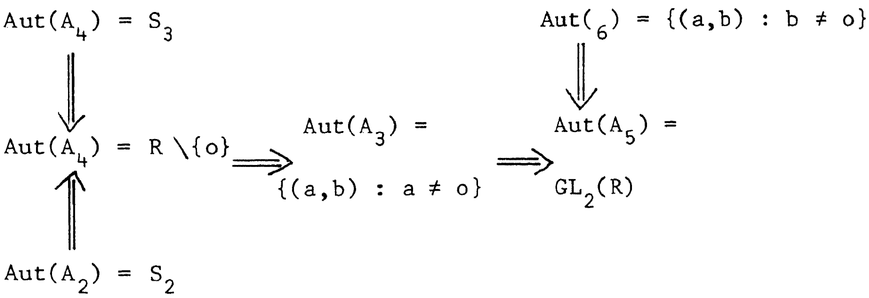


Figure 25

Here S_n is the permutation group on n letters and $GL_m(R)$ is the group of real invertible m x m matrices. The multiplication rules of $Aut(A_3)$ and $(Aut(A_6)$ are respectively $(a,b)(c,d) = (ac, ad + bc^2)$, $(a,b)(c,d) = (c+ad, bd)$.

We see that as a rule during a contraction ($\rightarrow$)
(i) symmetrics are both gained and lost
(ii) the dimension of the symmetry group does not become less

3.5. Design versus generic symmetry

At least in a large number of cases, there seem to be two sources of symmetry. The first is what I like to call "generic symmetry" it is the symmetry which is possessed by almost all of the structures under consideration. As an example consider algebras of the form C[X]/f(X) where C denotes the complex numbers and f(X) is a polynomial of degree n. Almost all f(X) have n distinct roots and as a consequence almost all of these algebras have S_n as their automorphism group. The second source is what I like to call design symmetry which arises e.g. when parts of the structures under consideration are very carefully arranged in such a way that a large symmetry group arises. For instance if precisely two roots of f(X) are equal and all others are different from each other and from this double root, then the automorphism group picks up a factor $R-\{0\}$ but not all roots are of the same kind anymore and the generic symmetry group drops to S_{n-2} (the permutations of the (n-2) unequal single roots). It seems to me that the way the symmetry groups can change often can be understood systematically in these terms. During contraction ($\Rightarrow$) the generic symmetry group tends to become smaller and the design symmetry group larger or, equivalently, during a deformation ($\Leftarrow$) the generic symmetry group becomes larger and the design symmetry group smaller (i.e. gets broken). This last phenomenon was of course the subject matter of section 2 above.

4. CONSEQUENCES OF THE PRESENCE OF SYMMETRY

Quite generally the presence of symmetry in a (mathematical) problem can have enormous influence and it can greatly facilitate solving a problem. We shall see this when examining bifurcation phenomena in the presence of symmetry in the next section and we have already seen examples in the previous section. Here we describe some more material around this theme.

4.1. Symmetric problems and their solutions

A first question to examine is whether symmetric problems necessarily have symmetric solutions. This is discussed in considerable detail by W.C. Waterhouse in [40] who calls the principle that this be the case the Purkiss principle. Here are some of his examples where the principle holds.

- of all rectangles with a given perimeter the square has the largest area.
- Take four positive numbers whose product is 16. Then their sum is least when all numbers are equal.
- For a given mean $\bar{x} = n^{-1}(x_1 + \ldots + x_n)$ the value of $x_1^2 + \ldots + x_n^2$ is least when all x_i are equal.

It is clear that statements to the effect that under certain circumstances the Purkiss principle holds are precisely the desired sort of mathematical explanations of why nature likes symmetric solutions (Cf. the introduction).

There are also quite simple counter examples to the Purkiss principle. For instance one from Buniakovsky: find the minimum of $f(x,y) = (x^2 + (y-1)^2)((x-1)^2 + y^2)$. This is symmetric in x and y. The two solutions, however, are (1,0) and (0,1). Another example (from [20]) which I like very much is the following. Consider four towns located on the corners of a square. What is the shortest road system that joins these four towns. It is not very difficult to see that there are two solutions which are depicted in Figure 26. The angle between the horizontal and the top left oblique segment in Figure 26 is 30°. This solution is better than the other obviously possible candidates: three edges of the square or the two diagonals.

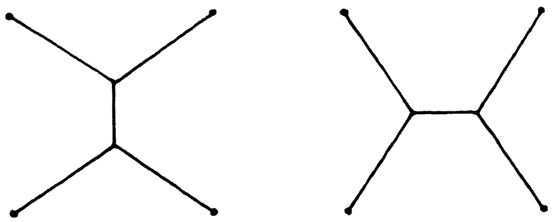


Figure 26

Note that in both counter examples the total set of solutions is symmetric under the full symmetry group of the problem but that the individual solutions (if there is more than one) are only invariant under an isotropy subgroup. That is, there is symmetry breaking in precisely the sense of section 2 above with as the manifold M the space of all solutions.

4.2. Extrema of symmetric functions

Here is a result that shows how strong the influence of the presence of non-trivial symmetry can be. The setting is that of section 2.3 above, i.e. a compact Lie group G (for instance a finite one) acting smoothly on a smooth manifold M. Let F be the set of all functions f on M which are invariant under G, i.e. such that f(gm) = F(m) for all $m \in M$, $g \in G$

Theorem (cf. e.g. [21]). If an orbit is isolated in its stratum it is critical for all $f \in F$ (i.e. df = 0 at all points of that orbit) and inversely if an orbit is critical for all $f \in F$ it is isolated in its stratum.

5. BIFURCATION IN THE PRESENCE OF SYMMETRY

5.1. General remarks and first examples

Bifurcation theory is concerned with how the set of solutions of a problem can change as a parameter varies. For a first introdution to bifurcation theory I refer the reader to my chapter "Bifurcation phenomena. A short introductory tu-

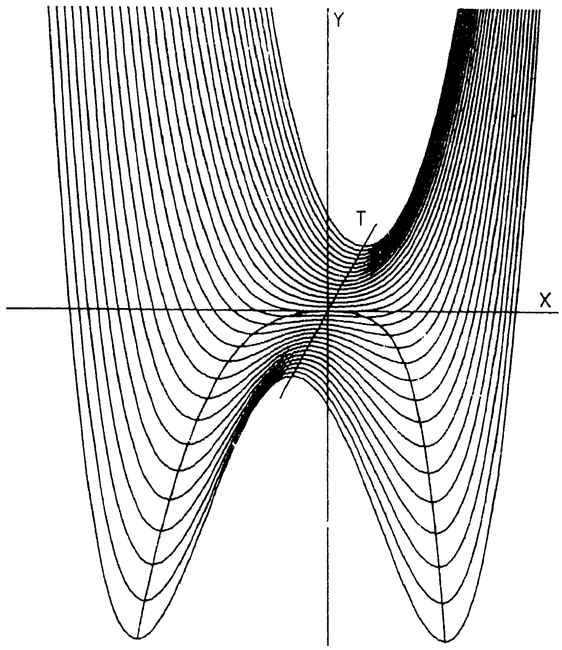


Figure 27

torial with examples" in this volume [15]. Two of the simplest examples of bifurcations are the pitchfork bifurcation depicted in Figure 27 where a minimum bifurcates into two minima and a maximum, and the Hopf bifurcation where a stable equilibrium point bifurcates into an oscillatory cycle. In the case of the bifurcating valley there is also obviously a kind of symmetry breaking involved and in the case of the Hopf bifurcation also seems to involve symmetry loss when viewed in space-time space, cf. fig 28 below. The continuous translations symmetry gets broken into a discrete group of translation symmetries. Quite generally it seems clear from the setting of sections 2 and 4 above that if a solution of a problem is symmetric with symmetry group G, and for parameter values $\lambda < \lambda_o$ there is (locally) a single solution, and if at λ_o this solution bifurcates into several, then the new solutions will have as symmetry group isotropy subgroups of G. This should severely

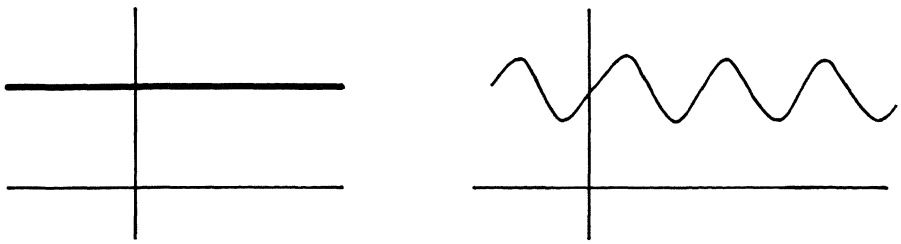


Figure 28

restrict the possible bifurcations. To see how this works, how the results of section 4 apply, and how representation theory can be used to advantage we first need a slightly more precise description of a little bit of bifurcation theory.

5.2. Microsynopsis of some bifurcation theory

Consider a map $G(\lambda,.) : B_1 \to B_2$ depending on a parameter λ (where λ may also denote a vector of parameters). Here the B_i are Banach spaces. At this stage it does not hurt to think of the B_i as finite dimensional spaces, say $B_1 = B_2 = R^2$ and to think of $(\lambda,.)$ as a map given by an expression like $(x^3 + y^3 + \lambda xy^2, x^2 + \lambda xy)$. We are interested in studying $G(\lambda,u) = 0$, $u \in B_1$, and in how the solution set of this equation changes as λ varies. In most of the interesting examples the B_i are suitable spaces of functions and $G(\lambda,.)$ is e.g. a differential operator depending smoothly on λ. An explicit example occurs below.

Suppose $G(\lambda_0, u_0) = 0$; consider the partial derivative $G_u(\lambda_0,u_0)$ at (λ_0, u_0). This is a linear map $B_1 \to B_2$. If this linear map is invertible then the implicit function theorem says that there is a differentiable mapping $\lambda \to u(\lambda)$ for λ near

λ_0 such that $G(\lambda, u(\lambda)) = 0$ and that locally (i.e. for (λ, u) close to (λ_0, u_0) this is the only solution. Thus for bifurcation phenomena we are interested in points where $G_u(\lambda_0, u_0)$ is not invertible. In the finite dimensional case of $B_1 = B_2 = R^n$ we are thus interested in cases where the $n \times n$ matrix of partial derivatives $G_u(\lambda_0, u_0)$ is not of full rank. If there is a symmetry group involved it is obvious how (generalised) results like those of section 4.2 above could, indeed will be important.

Assume that $G_u(\lambda_0, u_0)$ is a Fredholm operator of index zero, so that the kernel is finite dimensional and the range is closed of co-dimension equal to the dimension of the kernel (automatically the case if $B_1 = B_2 = R^n$.). In the simplest case $\dim N = 1$, where $N = \operatorname{Ker} G_u(\lambda_o, u_o)$. Let $B_1 \subset B_2$ and let $P : B_2 \to N$ be the projection on N and write $Q = Id - P$. Then we can rewrite the equation $G(\lambda,u) = 0$ as

$$QG(\lambda, v + \phi) = 0 \quad \text{and} \quad PG(\lambda, \phi) = 0$$

where $v = Pu$, $\phi = Qu$. Fix v, then $QG_u(\lambda_o, u_o)$ is an isomorphism, so by the same implicit function theorem used above we can find $\phi(\lambda, v)$ as a function of (v, λ) so that $Q(\lambda, v + \phi(\lambda, v)) = 0$ near (λ_o, v_o). Thus it remains to solve the so-called bifurcation equations

$$F(\lambda, v) \equiv PG(\lambda, v + \phi(\lambda, v)) : R \times N \to N, \; F(\lambda, v) = 0$$

In the simplest case ($\dim N = 1$) it readily follows that $F(\lambda,v)$ is of the form

$$F(\lambda,v) = a(\lambda - \lambda_o)\, v + \dots$$

so that in the non-degenerate case, the solution set near $(\lambda_o, 0)$ looks like two lines crossing each other vertically, a pitchfork bifurcation.

5.3. Equivariant bifurcation theory

Now suppose that there is a group of symmetries involved. I.e. there is a group H acting linearly on B_1 and B_2 and $G(\lambda,u)$ is equivariant which means that $G(\lambda,gu) = g\,G(\lambda,u)$ for all g G. Here is a general result [33, theorem 13].

<u>Theorem</u>. Let $G(\lambda,u) : B_1 \to B_2$ be analytic and equivariant w.r.t a compact group H. Let $G(\lambda_o, u_o) = 0$, $gu_o = u_o$ for all g H, and let $G_u(\lambda_o, u_o)$ be Fredholm of index zero with kernel N_o. Then N_o is invariant under H (i.e. gN_o N_o for all g H) and the $F(\lambda,v)$ are equivariant.

Now in many interesting bifurcation problems $\dim N > 1$. Then the simple analysis of 5.2 above does not apply. However, if there is symmetry it may easily happen that N is an irreducible representation of H and if we know which one (i.e. as a rule, if we know enough of the representation theory of H) this is just as good as the case dim N = 1.

We also know a priori that the bifurcating solutions will have symmetry groups which are isotropy subgroups of H acting on N. A very simple example of a bifurcation situation with symmetry (rotational symmetry in this case) is the one of a stiffish rubber bar with opposite forces acting on the two end points as shown in Figure 30. The corresponding bifurcation diagram is sketched of Figure 29.

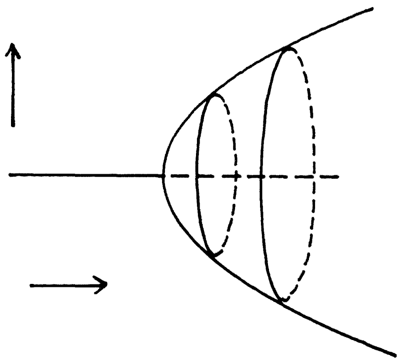


Figure 29

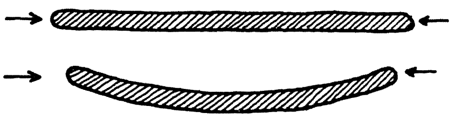


Figure 30

The idea is the following. Just as in case dim N = 1 we know that $F(\lambda,v)$ must start off with a term λv (if $\lambda_o = 0$), we know in the equivariant case that $F(\lambda,v)$ must start off with very specific terms which are determined by the representation of H, which is involved. In many cases degree considerations and the value of dim N rule out all but a few possible (known) representations which may make a case like dim N = t (which is usually totally intractable in the general case) quite easy to do in a symmetric case.

5.4. Example: Bénard convection [33, 34]; cf. also [9]

In case of the Bénard convection such an analysis has actually been carried out by Sattinger, loc. cit. The equations involved are the Boussinesq equations which are

$$\Delta u_k + \delta_{k3}\,\theta - \frac{\partial p}{\partial X_k} = P_r^{-1} \sum_j u_j \frac{\partial u_k}{\partial X_j} \quad , \; k = 1,\ 2,\ 3$$

$$\Delta\theta + Ru_3 = \sum_j u_j \frac{\partial \theta}{\partial X_j}$$

$$\sum_j \frac{\partial u_j}{\partial X_j} = 0$$

Here u_1, u_2, u_3 are the velocity components of a fluid element as functions of the co-ordinates X_1, X_2, X_3. The function θ is the temperature profile and P is the pressure. R denotes the Rayleigh number, P_1 the Prandtl number and Δ denotes the Laplacian

$$\frac{\partial^2}{\partial X_1^2} + \frac{\partial^2}{\partial X_2^2} + \frac{\partial^3}{\partial X_3^2}$$

and δ_{kl} the Kronecker delta. The bifurcation parameter λ is R. Thus in this case B_1 and B_2 are suitable spaces of 5-tuples of functions (u_1, u_2, u_3, θ, P) of three variables X_1, X_2, X_3, and it is then immediately clear how to write these equations in the form $G(\lambda, u) = 0$.

The group involved is E_2, the group of rigid motions of the plane which consists of all motions

$$g: \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \to \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}$$

where the matrix is orthogonal and hence of determinant ± 1.

The action of E_2 on the spaces of 5-tuples of functions B_1, B_2 is now given by

$$gf(X_1, X_2, X_3) = \begin{pmatrix} B & 0 \\ 0 & I \end{pmatrix} f\,(g^{-1}(X_1, X_2), X_3) \quad , \quad f \in B_i,\ g \in E_2$$

Here B is the 2 x 2 matrix which gives the rotation/reflection part of $g \in E_2$ and I is a 3×3 identity matrix. It is a small exercise to check that the Boussinesq equations are indeed equivariant w.r.t. this action. The reason is that the physics is independent of the observer.

In this case it turns out that dim N is infinite. However, under the assumption that we restrict our attention to solutions which are periodic with respect to a hexagonal lattice, cf. Figure 31, dim N becomes finite dimensional and an explicit bifurcation analysis can be carried out [33, 34, cf. also 9].

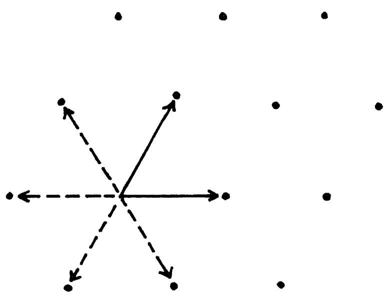


Figure 31

It turns out that the stability of the bifurcation solutions depends on two parameters in the manner depicted in Figure 32 below. This agrees with experimental data. Of course, this does not give us a complete mathematical description of the Benard convection. It remains to be shown that the E_2 symmetry has to break through a lattice. This is still an open problem, though it is hard to see how E_2 symmetry could get broken otherwise to a stable solution (cf. also [38, section II.1]). It also remains to analyse the relative stability of various lattice patterns with respect to each other. Cf. in this connection [31, 32].

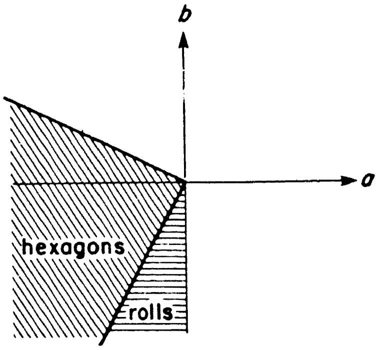


Figure 32

5.5. Drainage patterns. An example . Competing singularities

Let me conclude with a possible model for the formation of drainage patterns due to Thom [38], which I like particularly. Imagine a sandy slope on which a gentle rain is continuously falling. At the top small rivulets will form at random, these will merge and form larger rivulets further down etc., to form a pattern somewhat like the one depicted in Figure 33.

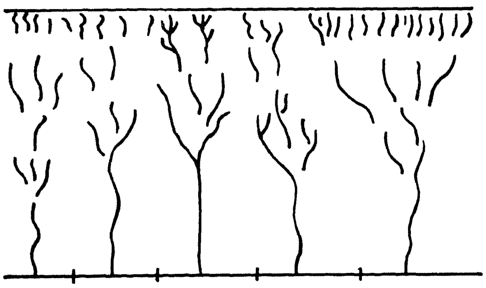


Figure 33

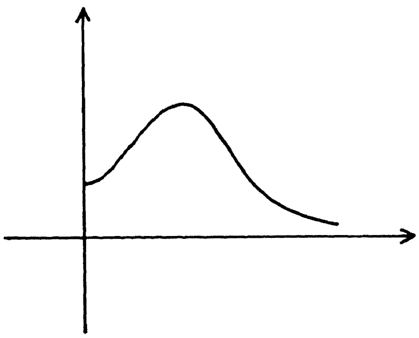


Figure 34

Let X_i denote the position of the i-th watershed near or at the bottom. Assume that the "erosion power" of a stream is proportial to the width of its basin. Then the position of the i-th watershed will move according to the differential equation

$$\dot{X}_i = C(X_i - X_{i-1}) - C(X_{i+1} - X_i) = C(2X_i - X_{i-1} - X_{i+1}) \tag{5.1}$$

Obviously equipartition ($X_i = \frac{1}{2}(X_{i-1} + X_{i+1})$) is a stationary solution. For an analysis of the stability of such a solution consider two streams at positions X and -X on R with the divide at u near zero. Then assuming that C also depends on X we find

$$\dot{u} = 2\,C(X)\,u + 2\,XC'(X)\,u + u^2\,(\dots\dots)$$

for small u. So that we will have stability if $C(X) + XC'(X) < 0$. Assuming that C(X) is somewhat like Figure 34, which is not unreasonable, one would expect a characteristic wave length for the pattern at bottom given by L = "smallest" X such that $C(X) + X C'(X) < 0$. The picture of Death Valley some pages back (Figure 4) is a nice example of just such a pattern with, apparently, a characteristic wave length. It remains of course to do a complete (bifurcation) analysis which leads from the random pattern at the top to the regular pattern at the bottom. The governing equation above is a discretisation of a certain (anti-) diffusion equation. Numerical experiments, [16], with such equations are encouraging but it seems that we are quite far from a completely satisfactory theory at the moment.